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# INDEXED LAWVERE THEORIES for LOCAL STATE

John Power

**ABSTRACT.** Monads for global state and local state have been used to provide semantics of programming languages for many years. There is a computationally natural presentation of an ordinary Lawvere theory that corresponds to the monad on *Set* for global state, inevitably called the Lawvere theory for global state. Here, we introduce a notion of indexed Lawvere theory and use it to give a Lawvere-style account of local state, extending the theorem for global state to local state. En route, we develop the notion of comodel of a Lawvere theory and exploit a universal characterisation of the category of worlds for local state. Ultimately, we give both syntactic and semantic characterisations of the operation *block* that allows one to move between worlds and use them to characterise the monad for local state.

## 1. Introduction

Monads have been used to model aspects of computational effects since the late 1980's [15, 16]. The equivalence between finitary monads and Lawvere theories was elaborated in the 1960's. So one might reasonably expect that it would have been clear from the outset that computational effects could be captured in either way. But, as a matter of historical fact, that is not true, as explained in [6].

The relationship between monads and Lawvere theories was an observation of no practical value in regard to computational effects until it was recognised, in 2001, that there is a computationally natural presentation of the Lawvere theory for global state [20]: we recall the details in Section 2. That recognition, followed quickly by corresponding recognition for all the other leading examples except for continuations, opened the way to the modelling of deeper aspects of computational effects [3, 4, 5, 19, 22, 23, 26]. This paper may be seen as part of the development. Specifically, we extend the notion of Lawvere theory to one of indexed Lawvere theory in order to characterise the monad for local state in a way that directly and naturally extends the Lawvere theory for global state: it is not as simple as asking for all models of an indexed Lawvere theory, but the notion of indexed Lawvere theory is fundamental.

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The monad on  $Set$  for global state, which was suggested to us by Peter O’Hearn, is  $T_S- = (- \times S)^S$ , where  $S$  is a set of states, typically given by  $V^{Loc}$  for a countable set  $V$  of values and a finite set  $Loc$  of locations. Corresponding to that, the countable Lawvere theory  $L_S$  for global state is freely generated by operations  $lookup : V \longrightarrow Loc$  and  $update : 1 \longrightarrow Loc \times V$  subject to computationally natural equations that we recall in Section 2. In order to extend from global to local state, one replaces  $Set$  by the functor category  $[Inj, Set]$ , where  $Inj$  is the category of finite sets and injections [17, 20, 21], or equivalently, the category of natural numbers and monomorphisms. The monad for local state, on  $[Inj, Set]$ , is

$$(T_{LS}X)_n = \left( \int^{m \in (n/Inj)} (Sm \times X_m) \right)^{Sn}$$

where  $Sn = V^n$  for a given set of values  $V$ ; we describe its behaviour on maps in Section 6. The idea of the monad is that in a state with  $n$  locations, a computation can create  $m - n$  new locations and return a value, e.g., a function, that depends on them, cf [11, 12]. In the case  $V = 1$  this reduces to the monad for local names in [28], which was introduced with an eye to the  $\pi$ -calculus.

The monad  $T_{LS}$  enriches over both the cartesian closed structure of  $[Inj, Set]$  and the convolution symmetric monoidal structure on  $[Inj, Set]$  generated by the sum of natural numbers. But with neither enrichment does the monad correspond to an enriched Lawvere theory for which we have been able to provide a computationally natural presentation. In [20], with an oversight corrected in [21], we did give an algebraic-like presentation of the monad  $T_{LS}$  extending that for global state, but it used an uncomfortable and uninvestigated hybrid of the two enrichments and it did not separate syntax from semantics. We managed somewhat better in [25], which at least includes a definition of indexed Lawvere theory, but even that paper had no presentation that separated syntax from semantics. That is what we provide here, gradually moving towards it through the course of the paper.

In order to extend global structure, such as that for global state, to local structure, one must first extend a Lawvere theory  $L$  to an indexed version of the same theory, i.e., if  $L$  has an operation  $f : a \longrightarrow b$ , then a local version also needs to have such an operation, but it must have it at each world, and it must be natural in the size of the world. For instance, for state, if one has  $n$  locations, there are  $n$  possible places at which one may either lookup or update. So we first seek a general construct that extends the operations of a global computational effect to become operations of an induced local effect. The canonical symmetric monoidal structure of  $Law$  allows us to do that:  $Inj$  is the free symmetric monoidal category for which the unit is the initial object (sometimes called a *coaffine* category [13, 18]) on 1;  $Law$  is another symmetric monoidal category with unit the initial object; and so the universal property allows us to extend any Lawvere theory  $L$  to an  $Inj$ -indexed such theory  $L_\otimes$ , uniformly allowing us to extend global operations to local operations. The details are in Section 3.

This example of an  $Inj$ -indexed Lawvere theory motivates an axiomatic definition of indexed Lawvere theory together with one of a model of an indexed Lawvere theory. Those definitions in turn motivate a subtle analysis of the relationship between  $D$ -indexed Lawvere theories for an arbitrary small category  $D$  and monads on  $[D, Set]$ , upon which we check what our abstract theory tells us in the example of state. This constitutes Section 4. Although we focus on state in this paper, the example of  $\pi$ -calculus may well also merit study in this light, cf [28]. Dependent

algebra seems likely to involve a notion of indexed Lawvere theory too, but it may be more complex than that we introduce here.

We next add an operation *block*: the central fact of locality is the capacity to create a new world. Semantically, *block* is modelled by a map of the form

$$M(n+1) \longrightarrow M(n)^V$$

uniform in  $n$  and subject to natural coherence conditions, where  $M(n)$  is a model of global state with  $n$  locations. So we first describe a general construct whose semantics extends a model  $M(n)$  at world  $n$  to a model of the form  $M(n)^V$  at world  $n+1$ . To do that, we observe that  $V$  is the final comodel of the ordinary Lawvere theory  $L_V$  for global state with value set  $V$  and one location. We then observe that exponentiating by the comodel  $V$ , the set  $M(n)^V$  has  $n+1$  *lookup* and *update* structures on it,  $n$  of them because  $M(n)$  has  $n$  such structures, with an additional one determined by the comodel structure of  $V$ . So we develop the idea of exponentiation by a comodel of a Lawvere theory in Section 5.

Having put  $M(n+1)$  and  $M(n)^V$  into the same category, we are in a position to analyse *block*. Some of the coherence conditions on *block* are equivalent to naturality. The remaining two conditions can be expressed as a unit condition and a symmetry condition. We describe the conditions, give semantics for *block*, and use it to characterise the monad  $T_{LS}$  for local state in Section 6.

Having modelled the semantics of *block*, we seek corresponding syntax. Specifically, we seek a construct that extends a Lawvere theory  $L$  for a global effect to include a syntactic correlate of *block*, subject to natural equations, for which a model is as above. We describe such syntax in Section 7 and end the paper by characterising the monad  $T_{LS}$  for local state in terms of the models of this extended notion of Lawvere theory.

As mentioned above, this paper completes the goal enunciated but not achieved in the conference paper [25]. The originality of this paper lies primarily in its development of syntax rather than semantics.

## 2. Global State

There are a few mildly different albeit equivalent definitions of Lawvere theory in the literature, so we spell out one definition for definiteness of the paper. Let  $Nat$  be the category of natural numbers and all maps between them. It has strictly associative finite coproducts given by the sum of natural numbers. Consequently the category  $Nat^{op}$  has strictly associative finite products.

**Definition 2.1.** A *Lawvere theory* consists of a small category  $L$  with strictly associative finite products and an identity-on-objects strict finite-product preserving functor  $J : Nat^{op} \longrightarrow L$ . The Lawvere theory is typically denoted by  $L$ . A *map* of Lawvere theories from  $L$  to  $L'$  is a functor from  $L$  to  $L'$  that commutes with  $J$  and  $J'$ : it necessarily strictly preserves finite products. A *model* of  $L$  in any category  $C$  with finite products is a finite product preserving functor  $M : L \longrightarrow C$ .

This definition forces the objects of a Lawvere theory  $L$  to be exactly the natural numbers. The definition of model specifically does not require strict preservation of finite products. For any Lawvere theory  $L$ , the models of  $L$  together with all natural transformations form a category  $Mod(L, C)$ . If  $C$  is locally presentable, the functor  $ev_1 : Mod(L, C) \longrightarrow C$  has a left adjoint, yielding a monad  $T_L$  on  $C$ . The category  $Mod(L, C)$  is then coherently equivalent to the category  $T_L-Alg$ . Taking

$C$  to be  $Set$ , the monads thus arising are exactly the finitary monads on  $Set$ . We denote the category of Lawvere theories by  $Law$ .

It is routine to generalise the definition of Lawvere theory to allow for countable arities rather than finite ones: one systematically replaces finiteness by countability, replacing  $Nat$  by the category  $\aleph_1$ , a skeleton of the category of countable (including the possibility of finite) sets and all functions between them. We denote the category of countable Lawvere theories by  $Law_c$ .

It is also routine to generalise the notion of Lawvere theory to that of enriched Lawvere theory, the case of primary interest in computing being enrichment in the cartesian closed category  $\omega Cpo$ , as explained in detail in [4].

The key example that motivated refinement of Moggi's modelling of computational effects by monads was given by global state [20]. Moggi modelled global state by use of the monad  $T_S - = (- \times S)^S$  on  $Set$  for a countable set  $S$  of states. This monad has countable rank, so necessarily corresponds to a countable Lawvere theory. But the Lawvere theory only became interesting when we could see a computationally natural presentation of it as follows [20]:

**Example 2.2.** Let  $V$  be a countable set of values, and let  $Loc$  be a finite set of locations. Put  $S = V^{Loc}$ , so  $S$  is the set of states with  $Loc$  locations taking values in  $V$ . The countable Lawvere theory  $L_S$  for state is freely generated by operations  $l : V \longrightarrow Loc$  and  $u : 1 \longrightarrow Loc \times V$  subject to equations expressed in terms of models in any category  $C$  with countable products as follows: a model in  $C$  consists of

- an object  $A$  of  $C$
- a lookup map  $l : A^V \longrightarrow A^{Loc}$ , and
- an update map  $u : A \longrightarrow A^{Loc \times V}$

subject to commutativity of two classes of diagrams. First, we have four interaction diagrams (NB: Paul André Mellies has just shown that the second axiom is redundant!) as follows:

$$\begin{array}{ccccc}
 A & \xrightarrow{u} & A^{Loc \times V} & \xrightarrow{\cong} & (A^V)^{Loc} \\
 \downarrow A^t & & & & \downarrow l^{Loc} \\
 A^{Loc} & \xleftarrow{A^\delta} & A^{Loc \times Loc} & \xleftarrow{\cong} & (A^{Loc})^{Loc}
 \end{array}$$

where  $\delta : Loc \longrightarrow Loc \times Loc$  and  $t : Loc \longrightarrow 1$  are the diagonal and terminal maps, and the lower unlabelled isomorphism matches the outer  $Loc$  of  $(A^{Loc})^{Loc}$  with the

first  $Loc$  of  $A^{Loc \times Loc}$ ,

$$\begin{array}{ccccc}
 (A^V)^V & \xrightarrow{l^V} & (A^{Loc})^V & \xrightarrow{\cong} & (A^V)^{Loc} \\
 \downarrow \cong & & & & \downarrow l^{Loc} \\
 A^{V \times V} & & & & (A^{Loc})^{Loc} \\
 \downarrow A^\delta & & & & \downarrow \cong \\
 A^V & \xrightarrow{l} & A^{Loc} & \xleftarrow{A^\delta} & A^{Loc \times Loc}
 \end{array}$$

where the unlabelled isomorphisms match the outer  $V$  of  $(A^V)^V$  with the first  $V$  of  $A^{V \times V}$  and similarly for  $Loc$ , cf [9],

$$\begin{array}{ccccc}
 A & \xrightarrow{u} & A^{Loc \times V} & \xrightarrow{u^{Loc \times V}} & (A^{Loc \times V})^{Loc \times V} \\
 \downarrow u & & & & \downarrow \cong \\
 A^{Loc \times V} & \xrightarrow{A^{Loc \times \pi_1}} & A^{Loc \times V \times V} & \xleftarrow{A^{\delta \times V \times V}} & A^{Loc \times Loc \times V \times V}
 \end{array}$$

where the unlabelled isomorphism matches the outside  $Loc$  with the first  $Loc$  and similarly for  $V$ , and

$$\begin{array}{ccccc}
 A^V & \xrightarrow{l} & A^L & \xrightarrow{u^{Loc}} & (A^{Loc \times V})^{Loc} \\
 \downarrow u^V & & & & \downarrow A^{\delta \times V} \\
 (A^{Loc \times V})^V & \xrightarrow{A^{Loc \times \delta}} & & & A^{Loc \times V}
 \end{array}$$

suppressing two isomorphisms. We also have three commutation diagrams as follows:

$$\begin{array}{ccccccc}
 (A^V)^V & \xrightarrow{l^V} & (A^L)^V & \xrightarrow{\cong} & (A^V)^{Loc} & \xrightarrow{l^{Loc}} & (A^{Loc})^{Loc} \\
 \downarrow s & & & & \downarrow s & & \\
 (A^V)^V & & & & & & (A^{Loc})^{Loc} \\
 \downarrow l^V & & & & & & \downarrow \\
 (A^{Loc})^V & \xrightarrow{\cong} & (A^V)^{Loc} & \xrightarrow{l^{Loc}} & (A^{Loc})^{Loc} & \longrightarrow & A^{Loc_2}
 \end{array}$$

where  $s$  signifies ‘swap’ maps and  $Loc_2$  denotes the set of ordered pairs of distinct elements of  $Loc$ , with the unlabelled maps both given by the same canonical map,

$$\begin{array}{ccccccc}
 A & \xrightarrow{u} & A^{Loc \times V} & \xrightarrow{u^{Loc \times V}} & (A^{Loc \times V})^{Loc \times V} & \xrightarrow{s} & (A^{Loc \times V})^{Loc \times V} \\
 \downarrow u & & & & & & \downarrow \\
 A^{Loc \times V} & \xrightarrow{u^{Loc \times V}} & (A^{Loc \times V})^{Loc \times V} & \longrightarrow & A^{Loc_2 \times V \times V} & & 
 \end{array}$$

where  $s$  again signifies a swap map and with the unlabelled maps again given by the same canonical map, and

$$\begin{array}{ccccccc}
 A^V & \xrightarrow{l} & A^{Loc} & \xrightarrow{u^{Loc}} & (A^{Loc \times V})^{Loc} & \xrightarrow{\cong} & (A^{Loc})^{Loc \times V} \\
 \downarrow u^V & & & & & & \downarrow \\
 (A^{Loc \times V})^V & \xrightarrow{\cong} & (A^V)^{Loc \times V} & \xrightarrow{l^{Loc \times V}} & (A^{Loc})^{Loc \times V} & \longrightarrow & A^{Loc_2 \times V}
 \end{array}$$

where, again, the unlabelled maps are given by the same canonical map.

The forgetful functor from  $Mod(L_S, Set)$  to  $Set$  induces the monad given by  $T_S - = (- \times S)^S$  for global state on  $Set$ . So, if one models *lookup* and *update* subject to natural computational equations, one derives the monad for global state, rather than needing to take the latter as a primitive [20]. This approach has the advantage of having the algebraic constructs that generate the computational effect, i.e., *lookup* and *update*, built into it [4].

### 3. Tensor Powers

The *tensor* of Lawvere theories, cf [2], takes a pair of Lawvere theories  $L$  and  $L'$ , takes all the operations and equations of each, and insists that each operation of  $L$  commutes with each operation of  $L'$ . More formally, it is defined as follows:

**Definition 3.1.** Given Lawvere theories  $L$  and  $L'$ , the Lawvere theory  $L \otimes L'$  is defined by the universal property of having maps of Lawvere theories from  $L$  and  $L'$  to  $L \otimes L'$ , with commutativity of all operations of  $L$  with respect to all operations of  $L'$ , i.e., given  $f : n \longrightarrow m$  in  $L$  and  $f' : n' \longrightarrow m'$  in  $L'$ , we demand commutativity of the diagram

$$\begin{array}{ccc}
 n \times n' & \xrightarrow{n \times f'} & n \times m' \\
 \downarrow f \times n' & & \downarrow f \times m' \\
 m \times n' & \xrightarrow{m \times f'} & m \times m'
 \end{array}$$

Tensor is characterised as follows [2, 4]:

**Theorem 3.2.** *Given Lawvere theories  $L$  and  $L'$ , the tensor  $L \otimes L'$  induces an equivalence of categories*

$$\text{Mod}(L \otimes L', \text{Set}) \simeq \text{Mod}(L, \text{Mod}(L', \text{Set}))$$

The tensor forms a symmetric monoidal structure on  $\text{Law}$ , with unit given by the initial object of  $\text{Law}$ , which is  $\text{Nat}^{op}$  together with the identity functor. Our primary use of this symmetric monoidal structure on  $\text{Law}$  is as a way to extend a Lawvere theory  $L$  to an  $n$ -fold version of it, modelling the idea of having  $n$  possible locations. It extends without fuss to countable Lawvere theories.

**Theorem 3.3.** [4] *Let  $L_S$  be the countable Lawvere theory for global state set  $S$  as in Example 2.2, and let  $L_{S'}$  be the countable Lawvere theory for global state set  $S'$ . Then the tensor product  $L_S \otimes L_{S'}$  is the countable Lawvere theory for global state set  $S \times S'$ .*

**Corollary 3.4.** Let  $L_V$  be the countable Lawvere theory for global state  $V$ , understood to be a set of values  $V$  with one location. Then the  $n$ -fold tensor product  $L_V^{(n)}$  is the countable Lawvere theory for global state  $V^n$ , i.e., the Lawvere theory for global state with value set  $V$  and  $n$  locations.

So, if one has  $n$  locations, the  $n$ -fold tensor product  $L_V^{(n)}$  of  $L_V$  describes those operations generated by the operations of global state, namely *lookup* and *update*, subject to their equations.

Observe that, although the category  $\text{Inj}$  of natural numbers and monomorphisms does not have finite coproducts, it has an initial object  $0$  that, together with the sum of natural numbers, equips  $\text{Inj}$  with a symmetric monoidal structure for which the unit is the initial object, aka it is *coaffine* [13, 18]. It is elementary to verify that it is the free symmetric monoidal category on  $1$  with unit the initial object. So the universal property of  $\text{Inj}$  allows us, starting with an arbitrary Lawvere theory  $L$  to generate a functor  $\text{Inj} \rightarrow \text{Law}$ .

#### 4. Indexed Lawvere Theories

Motivated by Section 3, we now introduce an axiomatic notion of indexed Lawvere theory. We shall make more sophisticated use of indexed Lawvere theories in analysing *block* in Section 7. We do not assert definitiveness of this definition: we really only have one substantial example of its use here, and although we could speak a little about  $\pi$ -calculus, cf [28], and we could write vague remarks about dependent algebra or about structures such as differential graded  $R$ -modules, cf [9], we have not substantially investigated them. When further such examples of indexed structures are studied from this perspective, the definition given here may well need to be refined. But it seems natural and it suits the purpose at hand, hence our introducing it.

**Definition 4.1.** For any small category  $D$ , a  *$D$ -indexed Lawvere theory* is a functor  $L : D \rightarrow \text{Law}$ . A *model*  $M$  of a  $D$ -indexed Lawvere theory  $L$  in a category  $C$  with finite products consists of, for every object  $d$  of  $D$ , a model

$$M_d : L_d \rightarrow C$$



of  $L_d$  in  $C$ , together with, for each map  $f : d \longrightarrow d'$  in  $D$ , a natural transformation

$$\begin{array}{ccc} L_d & \xrightarrow{M_d} & C \\ L_f \downarrow & \Downarrow M_f & \downarrow Id \\ L_{d'} & \xrightarrow{M_{d'}} & C \end{array}$$

respecting the composition and identities of  $D$ , i.e.,  $M_{id}$  is the identity and, given maps  $f : d \longrightarrow d'$  and  $f' : d' \longrightarrow d''$ , the natural transformation  $M_{f'f}$  is given by pasting  $M_{f'}$  on to  $M_f$ .

This duly extends to give a category  $Mod(L, C)$ : the arrows are  $D$ -indexed families of natural transformations that are coherent with respect to the  $M_f$ 's. For any  $D$ -indexed Lawvere theory  $L$  and category  $C$  with finite products, there is a forgetful functor

$$U : Mod(L, C) \longrightarrow [D, C]$$

taking a model  $M$  to the functor sending  $d$  to  $M_d(1)$ . For each  $d$ , recalling that  $L_d$  is an ordinary Lawvere theory, the functor  $U$  has a  $d$ -component of the form  $U_d : Mod(L_d, C) \longrightarrow C$ . For any locally presentable category  $C$ , each  $U_d$  has a left adjoint  $F_d$ , exhibiting  $Mod(L_d, C)$  as monadic over  $C$ . These left adjoints and this monadicity result can be combined as follows:

**Proposition 4.2.** For any  $D$ -indexed Lawvere theory  $L$  and any locally presentable category  $C$ , the forgetful functor  $U : Mod(L, C) \longrightarrow [D, C]$  has a left adjoint  $F$  given pointwise, i.e., for any functor  $X : D \longrightarrow C$ ,

$$(FX)_d = F_d X_d$$

Moreover, the adjunction exhibits  $Mod(L, C)$  as monadic over  $[D, C]$ .

**PROOF.** It follows from the study of ordinary Lawvere theories that an adjoint  $F_d$  exists for each  $d$ . One can routinely verify that such adjoints collectively satisfy the adjointness property for the forgetful functor  $U$ . Monadicity can be proved in several ways, one such proof being by Beck's monadicity theorem.  $\square$

Our leading class of examples of  $D$ -indexed Lawvere theories for this section has  $D = Inj$  and arises as follows:

**Definition 4.3.** Given any ordinary Lawvere theory  $L$ , we denote by  $L_\otimes$  the  $Inj$ -indexed Lawvere theory freely generated by the tensor structure of  $Law$  and the fact of  $Inj$  being the free symmetric monoidal category on 1 with unit the initial object. So  $(L_\otimes)n = L^{(n)}$ , the  $n$ -fold tensor product of  $L$ .

Applying Proposition 4.2 to the  $Inj$ -indexed Lawvere theories that thus arise allows us to conclude the following:

**Corollary 4.4.** Given any ordinary Lawvere theory  $L$  and any locally presentable category  $C$ , let  $T$  be the monad on  $C$  induced by the forgetful functor

$$Mod(L, C) \longrightarrow C$$

Then, the forgetful functor  $Mod(L_{\otimes}, C) \longrightarrow [Inj, C]$  exhibits  $Mod(L_{\otimes}, C)$  as monadic over  $[Inj, C]$  with monad  $T_{\otimes}$  given by

$$(T_{\otimes}X)_n = T^{(n)}X_n$$

where  $T^{(n)}$  is the monad on  $C$  induced by  $L^{(n)}$ .

All the above duly extends to countability and enrichment without fuss. So Corollary 3.4 yields our leading example of an *Inj*-indexed Lawvere theory and its models.

**Example 4.5.** By Corollary 3.4, if we let  $L_V$  be the countable Lawvere theory for global state with value set  $V$  and one location, the  $n$ -fold tensor  $L_V^{(n)}$  is the countable Lawvere theory for global state with value set  $V$  and  $n$  locations. There are  $n + 1$  canonical coprojections from  $L_V^{(n)}$  to  $L_V^{(n+1)}$  determined by the universal property of tensor as determined by Definition 3.1. Putting  $D = Inj$ , the universal property of *Inj* extends  $L_V$  to the functor

$$(L_V)_{\otimes} : Inj \longrightarrow Law$$

sending  $n$  to  $L_V^{(n)}$ , with behaviour of  $(L_V)_{\otimes}$  on maps determined by the canonical coprojections. A model  $M$  of  $(L_V)_{\otimes}$  in a category  $C$  with countable products consists of, for each  $n$ , a model  $M_n$  of  $L_V^{(n)}$ , i.e., a model of the countable Lawvere theory for state with value set  $V$  and  $n$  locations, in  $C$ , respecting the structural inclusions of  $L_V$  into each  $L_V^{(n)}$ .

Consider the significance of Corollary 4.4 to the example of global state. Applied to Example 4.5, it allows us to extend and reformulate Example 2.2 as follows:

**Example 4.6.** The category  $[Inj, Set]$  is a cartesian closed category. Given a set  $V$  of values, let  $V$  also denote the constant functor from *Inj* to *Set* at the object  $V$  of *Set*. Let *Loc* denote the object of  $[Inj, Set]$  given by the inclusion of *Inj* into *Set*. Using these conventions, Example 2.2 can be read as enriched in the cartesian closed category  $[Inj, Set]$  [20, 24]. The category  $Mod(L_S, [Inj, Set])$  of models of  $L_S$  qua  $[Inj, Set]$ -enriched countable Lawvere theory is equivalent to the category  $Mod((L_V)_{\otimes}, Set)$ . By Corollary 4.4, the induced monad  $T$  on  $[Inj, Set]$  is given by

$$(TX)_n = T_{V^n}X_n$$

i.e., the value of  $TX$  at  $n$  is given by applying the monad for global state with value set  $V$  and  $n$  locations to the set  $X_n$ .

## 5. Exponentiation by a Comodel

The technical development of the paper so far allows us to extend the operations and equations of any global effect to a local effect. But in order to account for locality, we also need to introduce a *block* operation. In order to do that, we introduce a general construction that extends an arbitrary model of the  $n$ -fold tensor product  $L^{(n)}$  of a Lawvere theory  $L$  to a model of  $L^{(n+1)}$ : for *block* makes an assignment of structure to a new variable. Our construction is based on the observation that exponentiation by a comodel of a Lawvere theory yields a model of the theory.

**Definition 5.1.** A *comodel* of a Lawvere theory  $L$  in a category  $C$  with finite coproducts is a model of  $L$  in  $C^{op}$ .

Comodels and natural transformations form a category  $Comod(L, C)$ , with a canonical forgetful functor to  $C$  determined by evaluation at 1.

**Theorem 5.2.** [27] *For any Lawvere theory  $L$ , the forgetful functor*

$$U : Comod(L, Set) \longrightarrow Set$$

*has a right adjoint, exhibiting  $Comod(L, Set)$  as comonadic over  $Set$ .*

PROOF. A proof and a construction are given by taking a dual of a standard construction for algebra in an axiomatic setting and simplifying it a little: replacing  $Set$  by  $Set^{op}$  requires care, but we only really need its limits and colimits, which we have [8].

The assertion of the existence of a right adjoint is equivalent to the assertion that the inclusion of  $Comod(L, Set)$  into  $[L^{op}, Set]$  has a right adjoint. For that, the key point is that for any natural number  $n$ , the functor  $n \times - : Set \longrightarrow Set$  sends a set  $X$  to the  $n$ -fold coproduct of copies of  $X$ , and it preserves limits of  $\omega$ -cochains.

Given a functor  $H : L^{op} \longrightarrow Set$ , there is, for each  $n$ , a canonical map of the form  $\phi_n : n \times H1 \longrightarrow H(n \times 1)$ . A map in  $L$  of the form  $f : n \longrightarrow 1$  is sent by  $H$  to a function of the form  $Hf : H1 \longrightarrow H(n \times 1)$ . So we need to pullback  $Hf$  along  $\phi_n$ . We need to do that simultaneously for all maps in  $L$ , yielding a limit  $H_11$  together with a function  $\sigma_1 : H_11 \longrightarrow H1$ . In general,  $H_1$  will not extend to a finite-coproduct preserving functor because of non-triviality of

$$n \times \sigma_1 : n \times H_11 \longrightarrow n \times H1$$

But one can continue inductively to build a cochain of length  $\omega$ : in defining  $H_21$ , one must also account for equalities in  $L$ , i.e., given  $f : n \longrightarrow 1$  and  $g_i : n_i \longrightarrow 1$  for  $1 \leq i \leq n$ , one must account for  $f(g_1, \dots, g_n)$ . But after defining  $H_21$ , it is routine to extend to all  $n$ . In the limit,  $H_\omega$  is functorial as one can define  $H_\omega f$  by using the limiting property of the cochain given by  $n \times \sigma_k$ . It is routine to verify that  $H_\omega$  is the cofree comodel on  $H$ .

For comonadicity, it is routine to verify the conditions for the dual of Beck's monadicity theorem, i.e., that  $U$  reflects isomorphisms and that split equalisers lift [1].  $\square$

Enrichment of Theorem 5.2 in a cartesian closed category is routine: we used the fact that the tensor of  $n$  with  $X$  had the universal property of a product. And the result generalises routinely to a countable version. So Theorem 5.2 generalises to categories such as  $Poset$  and  $\omega Cpo$ .

**Corollary 5.3.** For any Lawvere theory  $L$ , the category  $Comod(L, Set)$  has a terminal object, the final comodel.

Our leading example, which is also the leading example of [27], is given by global state:

**Example 5.4.** Let  $L_S$  be the countable Lawvere theory for global state, with  $S = V^{Loc}$  and where  $Loc$  is finite. The induced comonad on  $Set$  is given by  $(-)^S \times S$ ,

and the final comodel is therefore given by  $S = V^{Loc}$ . A fortiori, the final comodel of  $L_V$  in  $Set$  is given by  $V$ , with structural maps given by

$$loc = \delta : V \longrightarrow V \times V$$

and

$$upd = \pi_2 = t \times V : V \times V \longrightarrow 1 \times V \cong V$$

The category of comodels for the Lawvere theory  $L_S$  is equivalent to a category of arrays [27].

Given a comodel  $A : L^{op} \longrightarrow Set$  of an arbitrary Lawvere theory  $L$  in  $Set$  and given an arbitrary object  $X$  of a category  $C$  with products, letting  $X^-$  denote a product of copies of  $X$  in  $C$ , the composite

$$L \xrightarrow{A} Set^{op} \xrightarrow{X^-} C$$

preserves finite products. Thus  $X^{A1}$ , the  $A1$ -fold product in  $C$  of copies of  $X$ , inherits an  $L$ -model structure in  $C$  from the  $L$ -comodel  $A$  in  $Set$ . More generally, for any Lawvere theory  $L'$  and any model  $M : L' \longrightarrow C$  in a category  $C$  with products, the composite

$$L \xrightarrow{A} Set^{op} \xrightarrow{M^-} Mod(L', C)$$

preserves finite products. By Theorem 3.2, to give such a finite product preserving functor is equivalent to giving a model of  $L' \otimes L$ . This is natural in  $M$ , yielding the following:

**Proposition 5.5.** For any Lawvere theory  $L$  and any comodel  $A$  of  $L$  in  $Set$ , and for any Lawvere theory  $L'$  and any category  $C$  with products, exponentiation  $(-)^A$  induces a functor

$$(-)^A : Mod(L', C) \longrightarrow Mod(L' \otimes L, C)$$

**Corollary 5.6.** For any Lawvere theory  $L$ , let  $F$  be the final comodel of  $L$  in  $Set$ . Then, for any category  $C$  with products, exponentiation  $(-)^F$  induces a functor

$$(-)^F : Mod(L_{\otimes} n, C) \longrightarrow Mod(L_{\otimes}(n+1), C)$$

Moreover, this is natural in  $n$  as an object of  $Inj$ .

This duly generalises to the countable setting and also to the enriched setting. Our leading example, Example 4.6, extends as follows:

**Example 5.7.** Consider a model  $M$  of the  $Inj$ -indexed countable Lawvere theory  $(L_V)_{\otimes}$  in  $Set$ . Each  $(M_n 1)^V$  canonically possesses the structure of a model of  $L_V^{(n+1)}$ : it inherits  $n$   $L_V$ -structures from those of  $M_n 1$ , while the final  $L_V$ -comodel structure of  $V$  given by Example 5.4 induces a further  $L_V$ -structure on  $(M_n 1)^V$ .

The significance of this for us is that the semantics of  $block_n$  will be a map of  $L_V^{(n+1)}$ -models from  $M_{n+1}$  to  $M_n^V$ .

Observe that the above discussion is inherently semantic, i.e., it is about constructs on models. But the spirit of the idea of Lawvere theories is that we seek constructs on theories, and then consider a notion of model of that in any well-behaved semantic category  $C$  such as  $Set$ . In order to provide a construct at the level of theories, observe that, for any comodel  $A$  of a Lawvere theory  $L$ , the functor

$$(-)^A : \text{Mod}(L', C) \longrightarrow \text{Mod}(L' \otimes L, C)$$

of Proposition 5.5 makes the following diagram commute:

$$\begin{array}{ccc} \text{Mod}(L', C) & \xrightarrow{(-)^A} & \text{Mod}(L' \otimes L, C) \\ \downarrow ev_1 & & \downarrow ev_1 \\ C & \xrightarrow{(-)^{A1}} & C \end{array}$$

Letting  $C = \text{Set}$  for example, the functor  $(-)^{A1} : C \longrightarrow C$  is monadic [14, 23], with left adjoint given by  $A1 \times -$ , so with monad given by  $(A1 \times -)^{A1}$ , the monad for state with state set  $A1$ . So, letting  $L_{A1}$  denote the countable Lawvere theory for state with state set  $A1$ , it follows that  $\text{Mod}(L', C)$  is equivalent to  $\text{Mod}(L', \text{Mod}(L_{A1}, C))$ , which is in turn equivalent to  $\text{Mod}(L' \otimes L_{A1}, C)$ , and the diagram

$$\begin{array}{ccc} \text{Mod}(L', C) & \longrightarrow & \text{Mod}(L' \otimes L, C) \\ \uparrow \cong & & \downarrow ev_1 \\ \text{Mod}(L' \otimes L_{A1}, C) & \xrightarrow{ev_1} & C \end{array}$$

commutes. We can therefore conclude [1]:

**Theorem 5.8.** *For any Lawvere theory  $L$  and any comodel  $A$  of  $L$  in  $\text{Set}$ , and for any Lawvere theory  $L'$  and any locally presentable category  $C$ , there is a map of Lawvere theories  $f : L' \otimes L \longrightarrow L' \otimes L_{A1}$  such that the functor*

$$\text{Mod}(L' \otimes L_{A1}, C) \xrightarrow{\cong} \text{Mod}(L', C) \xrightarrow{(-)^A} \text{Mod}(L' \otimes L, C)$$

*is of the form  $\text{Mod}(f, C)$ .*

This can duly be iterated, yielding, for any  $n$ , a map of Lawvere theories of the form  $L' \otimes L^{(n)} \longrightarrow L' \otimes L_{A1^n}$ . We shall use this construct to elucidate *block* at the level of theories in Section 7.

We remark as an aside that, putting  $L' = \text{Nat}^{op}$  and  $C = \text{Set}$ , the functor

$$(-)^A : \text{Set} \longrightarrow \text{Mod}(L, \text{Set})$$

of Proposition 5.5 has a left adjoint: that left adjoint agrees with the central construct used in [23] to analyse structural operational semantics for computational effects.

## 6. Semantics for *block*

We have foreshadowed the semantics of *block* in Section 5. By Corollary 5.3, any Lawvere theory  $L$  has a final comodel  $F$ . By Corollary 5.6, for any category  $C$  with products, exponentiation  $(-)^F$  induces a functor

$$(-)^F : \text{Mod}(L_{\otimes n}, C) \longrightarrow \text{Mod}(L_{\otimes (n+1)}, C)$$

natural in  $n$ .

**Definition 6.1.** Given an arbitrary Lawvere theory  $L$  and a category  $C$  with products, a *block-algebra* in  $C$  consists of a model  $M$  of the *Inj*-indexed Lawvere theory  $L_\otimes$  in  $C$ , together with an indexed family of maps

$$block_n : M_{n+1} \longrightarrow M_n^F$$

in  $Mod(L_\otimes(n+1), C)$ , natural with respect to *Inj*, satisfying the following two equations:

$$\begin{array}{ccc} M_n(1) & \xrightarrow{(M_{inc})_1} & M_{n+1}(1) \\ & \searrow (M_n I)^t & \downarrow (block_n)_1 \\ & & (M_n 1)^F \end{array}$$
  

$$\begin{array}{ccc} M_{n+2} & \xrightarrow{M_{n+s}} & M_{n+2} \\ \downarrow block_{n+1} & & \downarrow block_{n+1} \\ M_{n+1}^F & & M_{n+1}^F \\ \downarrow block_n^F & & \downarrow block_n^F \\ (M_n^F)^F & \xrightarrow{s} & (M_n^F)^F \end{array}$$

where the second occurrence of  $s$  means swapping the two copies of  $F$  respectively.

The first axiom asserts that *block* only affects the newly created location, while the second asserts that in creating two new locations and assigning structure to them, one can do so in either order providing one remembers which location is being assigned which structure.

One can routinely form a category of *block*-algebras, the maps being maps of models of  $L_\otimes$  that respect the *block* structure. We denote the category by *block-Alg*.

**Theorem 6.2.** *If  $C$  is locally presentable, the composite forgetful functor*

$$block-Alg \longrightarrow [Inj, C]$$

*is monadic.*

**PROOF.** There are several proofs of this: a *block*-algebra is, by construction, given by adding operations subject to equations to a model of  $L_\otimes$ , and such models are in turn given by operations and equations on an object of  $[Inj, C]$ . Combining these facts, a *block*-algebra consists of an object of  $[Inj, C]$  together with operations subject to universally defined equations, and so the category of such is monadic over  $[Inj, C]$  as an instance of the general theory of [10].  $\square$

This routinely extends to countability and enrichment. We now check that, if  $L$  is the countable Lawvere theory for global state with value set  $V$  and one location, it yields the usual monad  $T_{LS}$  for local state on  $[Inj, Set]$ , following O'Hearn et al [11, 12, 17, 28] and agreeing with the monad of [20, 21].

The monad for local state, on  $[Inj, Set]$ , as suggested to us by Peter O'Hearn, is

$$(T_{LS}X)_n = \left( \int^{m\epsilon(n/I)} (Sm \times X_m) \right)^{S_n}$$

where  $S_n = V^n$  for a given set of values  $V$ , cf [11, 12]. If  $V = 1$  it reduces to the monad for local names in [28].

The behaviour of  $T_{LS}$  on injective maps  $f : n \rightarrow n'$  is as follows: decompose  $n'$  as the sum  $n + n''$ , note that  $S(p + n'') = Sp \times Sn''$ , and use covariance of  $X$ . So the map

$$\left( \int^{m\epsilon(n/I)} (Sm \times X_m) \right)^{S_n} \times Sn \times Sn'' \longrightarrow \int^{m''\epsilon((n+n'')/I)} (Sm'' \times X_{m''})$$

evaluates at  $Sn$ , then maps the  $m$ -th component of the first of the two coends into the  $(m + n'')$ -th component of the second, using the above isomorphism for  $S$  and functoriality of  $X$ .

In [20, 21], an algebraic-like signature generating  $T_{LS}$  was given by defining the corresponding category  $LS([Inj, Set])$  of algebras as follows, subject to some not entirely trivial rewriting: an algebra consists of

- an object  $A$  of  $[Inj, Set]$
- a lookup map with components  $l_n : n \times A(n)^V \rightarrow A(n)$
- an update map with components  $u_n : n \times A(n) \rightarrow A(n)^V$
- a block map with components  $b_n : A(n+1) \rightarrow A(n)^V$

subject to commutativity of seven interaction diagrams and six commutativity diagrams. The interaction diagrams consist of the four interaction diagrams for global state listed in Example 2.2 (but with the second now proved redundant by Paul André Mellies), together with

$$\begin{array}{ccc} A(n+1)^V & \xrightarrow{l_{n+1} \cdot (ev_{n+1} \times id)} & A(n+1) \\ \downarrow b_n^V & & \downarrow b_n \\ (A(n)^V)^V & \xrightarrow{A(n)^\delta} & A(n)^V \end{array}$$
  

$$\begin{array}{ccc} A(n+1) & \xrightarrow{u_{n+1} \cdot (ev_{n+1} \times id)} & A(n+1)^V \\ \downarrow b_n & & \downarrow b_n^V \\ A(n)^V & \xrightarrow{(A(n)^t)^V} & (A(n)^V)^V \end{array}$$

$$\begin{array}{ccc}
A(n) & \xrightarrow{A(\text{inc})} & A(n+1) \\
& \searrow A(n)_\varepsilon & \downarrow b_n \\
& & A(n)^V
\end{array}$$

The commutation diagrams are those for global state together with

$$\begin{array}{ccc}
A(n+2) & \xrightarrow{A(n+s)} & A(n+2) \\
\downarrow b_{n+1} & & \downarrow b_{n+1} \\
A(n+1)^V & & A(n+1)^V \\
\downarrow b_n^V & & \downarrow b_n^V \\
(A(n)^V)^V & \xrightarrow{s} & (A(n)^V)^V
\end{array}$$

and, for  $k$  less than or equal to  $n$ ,

$$\begin{array}{ccc}
A(n+1)^V & \xrightarrow{l_{n+1} \cdot (ev_k \times id)} & A(n+1) \\
\downarrow b_n^V & & \downarrow b_n \\
(A(n)^V)^V & \xrightarrow{l_n \cdot (ev_k \times id)} & A(n)^V
\end{array}$$
  

$$\begin{array}{ccc}
A(n+1) & \xrightarrow{u_{n+1} \cdot (ev_k \times id)} & A(n+1)^V \\
\downarrow b_n & & \downarrow b_n^V \\
A(n)^V & \xrightarrow{u_n \cdot (ev_k \times id)} & (A(n)^V)^V
\end{array}$$

One can observe directly the correspondence between these axioms and those we have developed: recalling that we refer to  $n$  copies of each of *lookup* and *update* rather than one  $n$ -parametrised version of each, we have already accounted for precisely the diagrams arising from global state in Example 4.6. So to give  $A$  together with  $l$  and  $u$  is equivalent to giving a model  $M$  of  $(L_V)_\otimes$  in  $Set$ . Now recall, from Example 5.4, that  $V$  is the final comodel of  $L_V$ . So, using Example 5.7, the first two interaction axioms together with the last two commutation axioms are equivalent to the assertion that  $b_n : M_{n+1} \longrightarrow M_n^V$  is a map in  $Mod(L^{(n+1)}, Set)$ . The remaining two axioms are exactly the axioms for *block* in Definition 6.1. This,



together with the characterisation of  $T_{LS}$  in [20, 21], allows us to conclude the following:

**Theorem 6.3.** *Taking  $L$  to be the countable Lawvere theory  $L_V$  for global state with value set  $V$  and one location, the category  $\text{block-Alg}$  is equivalent to  $LS([Inj, Set])$  and is thus monadic over  $[Inj, Set]$  with monad  $T_{LS}$  that for local state.*

## 7. Syntax for *block*

We end the paper by applying the notion of indexed Lawvere theory to give a syntactic description of *block*, its category of models characterising the monad  $T_{LS}$  on  $[Inj, Set]$  for local state. We first need to consider free strict symmetric monoidal categories with initial unit. These are an example of clubs and are thus among the easiest of structures for which to describe the free such: see [7] and the long succession of papers about clubs that followed it. We shall describe the one example we need after our definition.

**Definition 7.1.** Let  $Inj_2$  denote the free strict symmetric monoidal category on the arrow category  $\cdot \rightarrow \cdot$  with unit the initial object.

For concreteness, assume that the generating arrow for  $Inj_2$  is

$$x \xrightarrow{\gamma} y$$

An arbitrary object of  $Inj_2$  is a list of  $x$ 's and  $y$ 's, with the empty list acting as the unit. There are fully faithful functors from  $Inj$  into  $Inj_2$ , one of them,  $\iota_1$ , sending  $n$  to a list of  $n$  copies of  $x$ , the other,  $\iota_2$ , sending  $n$  to a list of  $n$  copies of  $y$ .

Observe that the map  $\gamma : x \rightarrow y$  induces a monoidal natural transformation  $Inj(\gamma) : \iota_1 \Rightarrow \iota_2$ . This, together with the fact that 0 is initial, makes the following diagrams commute:

$$\begin{array}{ccc} \iota_1(n) & \xrightarrow{\iota_1(n+!)} & \iota_1(n+1) \\ & \searrow \iota_1(n) \otimes ! & \downarrow \iota_1(n) \otimes \gamma \\ & & \iota_1(n) \otimes \iota_2(1) \end{array} \quad (1)$$

$$\begin{array}{ccc} \iota_1(n+2) & \xrightarrow{\iota_1(n+s)} & \iota_2(n+2) \\ \downarrow \iota_1(n+1) \otimes \gamma & & \downarrow \iota_1(n+1) \otimes \gamma \\ \iota_1(n+1) \otimes \iota_2(1) & & \iota_1(n+1) \otimes \iota_2(1) \\ \downarrow \iota_1(n) \otimes \gamma \otimes \iota_2(1) & & \downarrow \iota_1(n) \otimes \gamma \otimes \iota_2(1) \\ \iota_2(n) \otimes \iota_1(1) \otimes \iota_1(1) & \xrightarrow{\iota_2(n) \otimes s} & \iota_2(n) \otimes \iota_1(1) \otimes \iota_1(1) \end{array} \quad (2)$$

where the two occurrences of  $s$  mean the evident twist maps.

By Theorem 5, every Lawvere theory  $L$  together with a comodel  $A$  in  $Set$  generates a map of Lawvere theories  $A^* : L \longrightarrow L_{A1}$  and hence an  $Inj_2$ -indexed Lawvere theory  $L_{\otimes A}$ . Note that  $L_{\otimes A}(\iota_2(n)) = L_{A1^n}$ , the Lawvere theory for state with state set  $A1^n$ . We need to consider some, but not all, of the models of the  $Inj_2$ -indexed Lawvere theory  $L_{\otimes A}$ .

I do not currently see a way to avoid a condition on the models like this, even in the case of  $A$  being the final comodel. I suspect that that reflects the fact that this is a first paper introducing indexed Lawvere theories, indicating a lack of sophistication in the idea to date: I would not be surprised if there is an interesting, general construct that underlies this that is yet to be discovered; a detailed analysis of further examples, arising, for instance, from dependent algebra, may well uncover it.

**Definition 7.2.** Given a Lawvere theory  $L$  together with a comodel  $A$  of  $L$  in  $Set$ , an  $A$ -model of the free  $Inj_2$ -indexed Lawvere theory  $L_{\otimes A}$  on  $A^* : L \longrightarrow L_{A1}$  in a category  $C$  with products is a model  $M$  of  $L_{\otimes A}$  in  $C$  such that

$$M_{z \otimes \iota_2(n)} = (M_z -)^{A1^n} : L_{\otimes A}(z \otimes \iota_2(n)) \longrightarrow C$$

naturally in  $z \in Inj_2$  and strong symmetric monoidal naturally in  $n \in Inj$ .

The definition of  $A$ -model extends routinely to yield a category  $AMod(L, C)$ , the arrows being the maps of models of  $L_{\otimes A}$  that respect the  $A$ -structure.

**Theorem 7.3.** *If  $C$  is locally presentable, and  $F$  is the final comodel of a Lawvere theory  $L$ , the categories  $FMod(L, C)$  and  $block-Alg$  are coherently equivalent.*

PROOF. The proof is given by tedious checking of a mass of detail. The heart of it is the fact that the coherence conditions in Definition 7.2 mean that an  $A$ -model of the free  $Inj_2$ -indexed Lawvere theory  $L_{\otimes A}$  on  $A^* : L \longrightarrow L_{A1}$  in  $C$  is given by

- a model  $M$  of  $L_{\otimes}$  in  $C$
- for each  $n$ , a map of  $L_{\otimes}(n+1)$ -models

$$M_{n+1} \longrightarrow M_n^F$$

- subject to naturality in  $n \in Inj$
- subject to commutativity of the images of (1) and (2).

Considerable calculation shows that preservation of the commutativity of (1) and (2), together with the naturality conditions, yields preservation of all commutative diagrams in  $Inj_2$ , while agreeing with the two commutativity conditions in the definition of  $block$ -algebra.  $\square$

The above duly extends to countability and enriches. Recall that if  $L_V$  is the countable Lawvere theory for global state with one location and with value set  $V$ , the final comodel of  $L_V$  is given by  $V$ .

**Corollary 7.4.** Let  $L_V$  be the countable Lawvere theory for global state with one location and value set  $V$ . Then  $VMod(L_V, Set)$  is monadic over  $[Inj, Set]$  with monad  $T_{LS}$ , the monad for local state.

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